Unit 6: Matrices Revisited and Game Theory (YEAH!) Name Key

Section 1: Markov Chains

A **Markov Chain** is a process that arises in problems that involve a finite number of events that change over time.

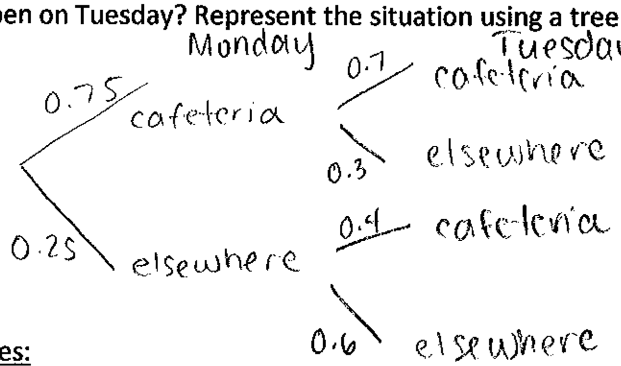
Consider the following situation:

A food service director at a high school conducted a survey in hopes of predicting the number of students who will eat in the cafeteria in the future. The results are as follows:

- If a student eats in the cafeteria on a given day, the probability that he or she will eat there again the next day is 70%.
- If a student does not eat in the cafeteria on a given day, the probability that he or she will eat in the cafeteria the next day is 40%.

Suppose that on Monday, 75% of the students ate in the cafeteria and 25% ate elsewhere. What can be expected to happen on Tuesday? Represent the situation using a tree diagram below:

Tuesday
0.625 · cafeteria
0.375 · elsewhere



$= 0.75(0.7) = .525^{*} = P(c \,\&\, c)$

$= 0.75(0.3) = .225 = P(c \,\&\, e)$

$= 0.25(0.4) = .100^{*}\ P(e \,\&\, c)$

$= 0.25(0.6) = .150 = P(e \,\&\, e)$

Transition Matrices:

The Monday student data are called the **initial distribution** of the student body and can be represented by a row (or **initial state**) matrix, D_0. Write the row matrix D_0 below.

$$D_0 = \begin{array}{cc} C & E \\ \end{array}$$
$$D_0 = \begin{bmatrix} 0.75 & 0.25 \end{bmatrix}$$

When we move from one state to another, we call it a transition. So, the data about how students choose to eat from one day to the next can be written in a matrix called a

transition matrix, *T*. Write matrix *T* below.

$$T = \begin{array}{c|cc} & C & E \\ \hline C & 0.7 & 0.3 \\ E & 0.4 & 0.6 \end{array}$$

All of the entries in these matrices are probabilities so their values will always be between 0 and 1 inclusive. These matrices are also always square matrices where the sum of the probabilities in any row is always 1.

1. Calculate the product of the matrix D_0 and matrix *T*.

$$\overset{D_0}{\begin{bmatrix} 0.75 & 0.25 \end{bmatrix}} \cdot \overset{T}{\begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}} = \begin{bmatrix} 0.75(0.7)+0.25(0.4) & 0.75(0.3)+0.25(0.6 \end{bmatrix}$$
$$= \begin{bmatrix} 0.625 & 0.375 \end{bmatrix}$$

2. Compare these calculations to your tree diagram on the last page.

$D_1 = \begin{bmatrix} 0.625 & 0.375 \end{bmatrix}$ 0.625 - cafeteria
0.375 - elsewhere

3. The values in the resulting row matrix can be interpreted as the portion of students who eat in the cafeteria and who eat elsewhere on Tuesday. Label your new matrix D_1.
4. To make a prediction for Wednesday, what two matrices should we multiply? $D_1 \cdot T$

$$\begin{bmatrix} 0.625 & 0.375 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

5. Determine the values in our new matrix D_2 (our predicted number of students eating in the cafeteria on Wednesday).

$$\begin{bmatrix} 0.625(0.7)+0.375(0.4) & 0.375(0.3)+0.375(0.6) \end{bmatrix} = D_2 = \begin{array}{cc} C & E \\ \left[0.5875 \right. & \left. 0.4125 \right] \end{array}$$

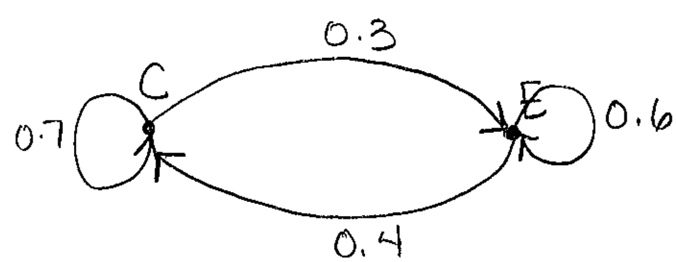
6. Consider how D_2 was calculated: $D_2 = D_1T$, but $D_1 = D_0T$. Use substitution to show a new calculation for D_2.

$$D_2 = (D_0T)T = D_0T^2$$

7. Based on this 'new' formula, how could I find the distribution for lunch in the cafeteria on Friday (day 4)?

$$D_4 = D_0T^4 = \begin{bmatrix} 0.75 & 0.25 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}^4$$
$$= \begin{bmatrix} 0.572875 & 0.427125 \end{bmatrix}$$

8. Instead of using transition matrices, we can also show the movement of students in a weighted digraph called a **transition digraph** (or **state diagram**). Show a transition digraph for our cafeteria statistics below.



Section 2: Game Theory (Part 1)

Game Theory is a branch of mathematics that uses mathematical tools to study when two or more individuals try to control the course of events, often resulting in conflict.

In this lesson, you will explore some examples of games with two players and use matrices to determine the best strategy for each player to choose.

Explore This…

Person A and Person B are concealing (showing) a penny with either heads or tails turned upward. They display their pennies simultaneously.

- A wins three pennies from B if both coins are heads.
- B wins two pennies from A if both are tails, and one penny from A if the coins don't match.

Play this game with your table partner. Write down some strategies for Person A and Person B.

Person A	Person B
Varies	Varies

Who ends up having the better deal? Person A or Person B? Why?

Person B → always display tails and can't lose!

A game in which the best strategy for both players is to pursue the same strategy every time is called Strictly determined. A good way to organize these rather boring 'games' is by using matrices.

Matrix Representation

Write a matrix that presents Person A's view of the game. This type of matrix is called a payoff matrix.

Person B

$$\text{Person A}\quad \begin{array}{c c} & \begin{array}{cc} H & T \end{array} \\ \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} \end{array}$$

This matrix is easy to follow if you are Person A, but the entries are just the opposite if you are Person B. Write a second matrix from Person B's view of the game.

Person B

$$\text{Person A} \quad \begin{array}{c c} & \begin{array}{cc} H & T \end{array} \\ \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix} \end{array}$$

Consider the game from Person A's point of view. Person A does not want to lose any more money than necessary, so he analyzes his strategies from the standpoint of his losses. If he displays heads, the worst he can do is to ___lose 1 ¢___. If he displays tails, the worst he could do is ___lose 2 ¢___.. What should Person A display? ___heads___

Person A's analysis can be related to the payoff matrix by writing the worst possible outcome of each strategy to the right of the row that represents it. The worst possible outcome of each strategy is the smallest value of each row, referred to as the ___row minimum___.

B

$$A \quad \begin{array}{c c} & \begin{array}{cc} H & T \end{array} \\ \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{Row minimums} \\ -1 \\ -2 \end{array}$$

Person A's best strategy is to select the option that produces the largest of these minimums (or the "best of the worst"). Because this value is the largest of the smallest row values, it is called the ___maximin___.

Because Person B's point of view is exactly opposite, she views the minimums as maximums and vice-versa. Therefore, her best strategy is the one associated with the smallest of the largest values, called the ___minimax___.

B

$$A \quad \begin{array}{c c} & \begin{array}{cc} H & T \end{array} \\ \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} \\ \text{column max} & \begin{array}{cc} 3 & -1 \end{array} \end{array}$$

Both Person A and Person B have a maximin or minimax that are the same value (called a **saddle point**). This is the identifying characteristic of **strictly determined** games. If the value selected by the two players is NOT the same, then the game is not strictly determined (we will investigate these in the next lesson!).

Games with More than 2 Strategies:

When players have more than two strategies, a game is harder to analyze. It is helpful to eliminate strategies that are **dominated** by other strategies.

For example: Two pizza places, A and B, are considering 4 strategies:

1. Running no specials
2. Offering a free mini-pizza with the purchase of a large pizza
3. Offering a free medium pizza with the purchase of a large pizza
4. Offering a free drink with any pizza purchase

A market study estimates the gain in dollars per week to A over B according to the following payoff matrix:

		Place B				
		No Special	*Mini*	*Medium*	*Drink*	row minimums
	~~*No Special*~~	~~200~~	~~−400~~	~~−300~~	~~−600~~	−600
Place A	*Mini*	500	100	200	600	100
	~~*Medium*~~	~~−400~~	~~−100~~	~~−200~~	~~−300~~	−300
	Drink	300	0	400	−200	−200
	column maximums	500	100	400	600	

What should the managers of the two restaurants do?

Both should do the mini-pizza special, but Place A will gain \$100/wk over Place B.

Strategies:

- Suppose you are the manager of place A and examine the first two rows carefully. Notice that the first row of the matrix is dominated by the second because each number in row 2 is greater than or equal to its corresponding number in row 1. Row 1 can be eliminated by drawing a line through it. Similarly, the second row dominates the third, and so the third row can also be eliminated.
- Now suppose you are the manager of place B. Because all the payoffs to B are opposites of the payoffs to A, a column is dominated if all its entries are greater than or equal to, rather than less than, those of another column. Which two columns can be eliminated, given this information?

  column 1 and column 3

- Write in the row minimums and column maximums. Is there a saddle point? What is the best strategy? Who comes out ahead?

  Saddle point = \$100
  Mini-pizza for both
  Place A comes out \$100/wk ahead

Section 3: Game Theory (Part 2)

Suppose the game from the last section is changed a bit after Person A realizes they will ALWAYS lose if Person B plays rationally. The new rules are:

- If both coins are heads, Person A will win 4 pennies.
- If both coins are tails, Person A will win 1 penny.
- If Person A shows heads and Person B shows tails, Person A loses 2 pennies.
- If Person A shows tails and Person B shows heads, Person A loses 3 pennies.

Show the new payoff matrix for this new game. Write in the row minimums and column maximums.

Person B

$$\text{Person A} \quad \begin{array}{c} \\ H \\ T \end{array} \begin{array}{c} \begin{array}{cc} H & T \end{array} \\ \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{c} \text{Row mins:} \\ -2 \\ -3 \end{array}$$

Column maxs: 4 1

Do the minimax and the maximin match? What does this mean for this game? No – not strictly determined

Play this game with your table partner. What is the best strategy for Person A?

varies

Using a probability tree, show the probabilities for all of the possible outcomes if Person A and Person B decided to flip their coins instead of strategically picking heads or tails.

Person A → Person B

0.5 H → 0.5 H = P(A & B heads) = 0.5(0.5) = 0.25

0.5 H → 0.5 T = P (A heads & B tails) = 0.5(0.5) = 0.25

0.5 T → 0.5 H = P (A tails & B heads) = 0.5(0.5) = 0.25

0.5 T → 0.5 T = P (A & B tails) = 0.5(0.5) = 0.25

The probability distribution for Person A's winnings for this case can be written in a table:

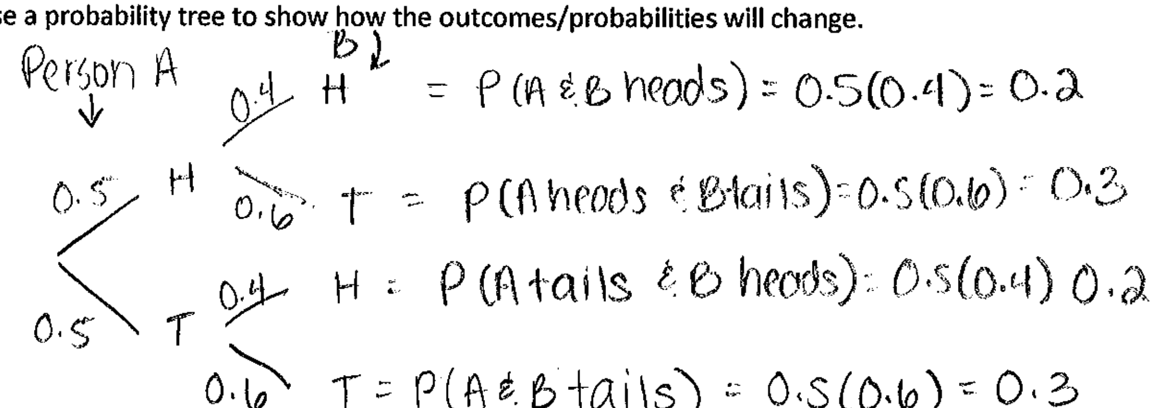
Outcome	HH	HT	TH	TT
Probability	0.25	0.25	0.25	0.25
Amount won	4	-2	-3	1

What is Person A's expected payoff? $0.25(4)+0.25(-2)+0.25(-3)+0.25(1)=0$

What is Person B's expected payoff? $0.25(-4)+0.25(2)+0.25(3)+0.25(-1)=0$

If both players display heads and tails in equal proportions, the game is considered fair because their expected payoffs are equal.

Suppose Player B decides to play heads 40% of the time, while Person A decides to continue flipping the coin. Use a probability tree to show how the outcomes/probabilities will change.



The probability distribution for Person A's winnings for this case can be written in a table:

Outcome	HH	HT	TH	TT
Probability	0.2	0.3	0.2	0.3
Amount won	4	-2	-3	1

What is Person A's expected payoff in this situation? $0.2(4)+0.3(-2)+0.2(-3)+0.3(1)=-0.1$

Does one player have an advantage over the other in this scenario? What does that mean about the game now?

Player B has a slight advantage (gains 1¢ for every 10 plays). The game is no longer considered "fair."

Now reconsider the game from Person A's point of view, and suppose that Person B plays heads every time while Person A continues to flip the coin. Find the expected payoff for Person A. Pason A payoff =

P(both heads) = 0.5 (1) = 0.5
P (H,T) = 0.5 (0) = 0
P (T, H) = 0.5 (1) = 0.5
P (T,T) = 0.5 (0) = 0

$0.5(4)+0(-2)+0.5(-3)+0(1) = \boxed{0.5}$

If Person B decides to play tails every time while Person A continues to flip the coin, what would the expected payoff be for Person A?

P(HH) = 0.5(0) = 0
P(HT) = 0.5(1) = 0.5
P(TH) = 0.5(0) = 0
P(TT) = 0.5(1) = 0.5

Person A payoff =
$0(4)+0.5(-2)+0(-3)+0.5(1) = \boxed{-0.5}$

With your table partner, explore these scenarios:

- Person A displays heads 60% of the time while Person B always displays heads.

  $0.6(4)+0(-2)+0.4(-3)+0(1) = \boxed{1.2}$

- Person A displays heads 60% of the time while Person B always displays tails.

  $0(4)+0.6(-2)+0(-3)+0.4(1) = \boxed{-0.8}$

- If the probability Person A will display heads is *p*, his expected winnings per play, if Person B displays all heads or tails is what? (Show as a product of a row matrix with your payoff matrix from the beginning of this section). Find the value of *p* by setting the two expected payoffs equal to each other. What is Person A's best strategy?

$$\begin{bmatrix} p & 1-p \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 4p-(3-3p) & -2p+(1-p) \end{bmatrix}$$

$4p-3+3p = -2p+1-p$

$7p-3 = -3p+1$

$+3p \quad +3 \quad +3p \quad +3$

$\boxed{\text{Person A should display H 2/5 times}}$ $10p = 4$ $\boxed{p = \frac{2}{5}} = .4$ $\boxed{1-p = \frac{3}{5}} = .6$

- Person B's best strategy can be determined in a similar way. Call the probability that she displays heads *q*. Because she is the 'column' player, multiply the payoff matrix by a column matrix to obtain her expected payoffs if Person A plays either all heads or all tails.

$$\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} q \\ 1-q \end{bmatrix} = \begin{bmatrix} 4q-2+2q \\ -3q+1-q \end{bmatrix}$$

$6q-2 = -4q+1$

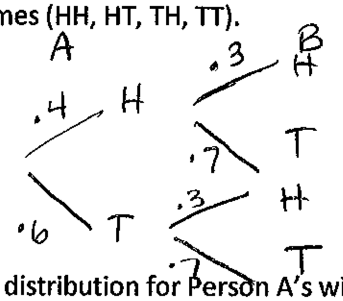
$+4q \quad +2 \quad +4q \quad +2$

$10q = 3$

$q = \frac{3}{10} = .3 \quad 1-q = \frac{7}{10} = .7$

$\boxed{\text{Person B should display heads 3/10 times}}$

Using the "best strategies" from above, use a probability tree to show the probabilities of the four possible outcomes (HH, HT, TH, TT).



$P(HH) = 0.4(0.3) = .12$

$P(HT) = 0.4(0.7) = .28$

$P(TH) = 0.6(0.3) = .18$

$P(TT) = 0.6(0.7) = .42$

The probability distribution for Person A's winnings for this case can be written in a table:

Outcome	HH	HT	TH	TT
Probability	.12	.28	.18	.42
Amount won	4	-2	-3	1

What is Person A's expected payoff?

$$.12(4) + .28(-2) + .18(-3) + .42(1) = \boxed{-0.2}$$

If both players pursue the best strategies, who is favored? By how much?

Player B is favored (shewill win 2 pennies out of every 10 plays).