Unit 6: Homework Packet Name Key

Section 1: Markov Chains

Use the following transition matrix for questions 1 and 2 below: $T = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$

1. For parts a – c, use the initial distribution $D_0 = [0.75 \;\; 0.25]$.

   a. Find the distribution of students eating and not eating in the cafeteria each day for the first week of school.

   Tues. $D_1 = [.625 \quad .375]$ Thu. $D_3 = [.57625 \quad .42375]$

   Wed. $D_2 : [.5875 \quad .4125]$ Fri. $D_4 = [.572875 \quad .427125]$

   b. Find the distribution of students eating and not eating in the cafeteria after 2 weeks (10 school days) have passed. Repeat for 3 weeks (15 days).

   10 days = $D_9 = [.5714320863 \quad .4285679138]$

   15 days = $D_{14} = [.57142858 \quad .42857142]$

   c. Based on your computations in parts a and b, what do you notice?

   In the long run, about 57% of students could be expected to eat in the cafeteria and 43% elsewhere.

2. Suppose the entire student body eats in the cafeteria on the first day of school. The initial distribution in this case is $D_0 = [1 \;\; 0]$. Repeat parts a and b of exercise 1 for this distribution. After several weeks, what percentage of students will be eating in the cafeteria?

   $D_1 = [.7 \quad .3]$ $D_3 = [.583 \quad .417]$

   $D_2 : [.61 \quad .39]$ $D_4 = [.5749 \quad .4251]$

   $D_9 = [.571437007 \quad .428562993]$

   $D_{14} = [.5714285919 \quad .4285714081]$

   ≈ 57% of students would be expected to eat in the cafeteria.

3. Which of the matrices below could be Markov transition matrices? For the matrices that could not be transition matrices, explain why not. 2×3

a. $\begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.6 \end{bmatrix}$ b. $\begin{bmatrix} \boxed{1.2} & -4 \\ 1 & 0 \end{bmatrix}$ c. $\begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.3 \end{bmatrix}$ d. $\begin{bmatrix} 0.75 & 0.25 \\ 1 & 0 \end{bmatrix}$

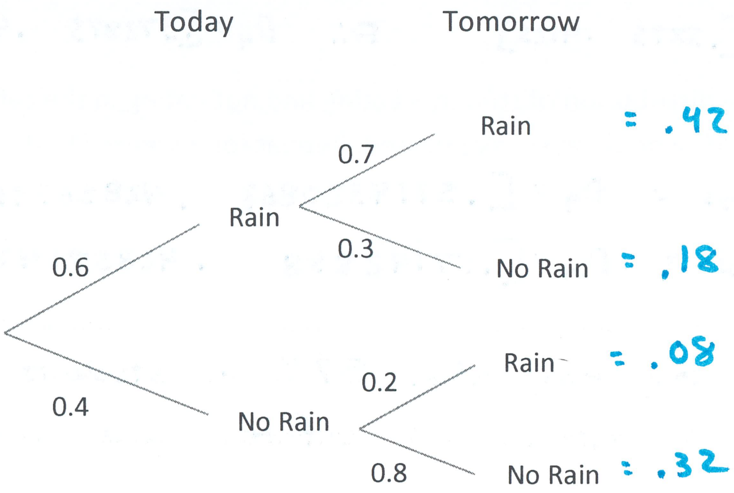
No, the sum in row 2 is > 1.

No, each element must be between 0 and 1.

No, not a square matrix.

Yes!

4. There is a 60% chance of rain today. It is known that tomorrow's weather depends on today's according to the probabilities shown in the following tree diagram.



a. What is the probability it will rain tomorrow if it rains today? .42

b. What is the probability it will rain tomorrow if it doesn't rain today? .08

c. Write an initial-state matrix that represents the weather forecast for today.

$$D_0 = \begin{array}{c} \begin{array}{cc} R & N \end{array} \\ \begin{bmatrix} .6 & .4 \end{bmatrix} \end{array}$$

d. Write a transition matrix that represents the transition probabilities shown in the tree diagram.

$$T = \begin{array}{c|cc} & R & N \\ \hline R & .7 & .3 \\ N & .2 & .8 \end{array}$$

e. Calculate the forecast for 1 week (7 days) from now.

$$D_0 T^6 = [.403125 \quad .596875]$$

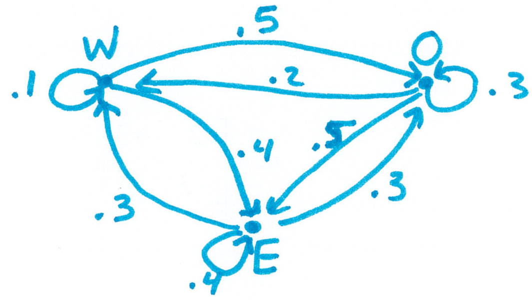
f. In the long run, for what percentage of days will it rain?

≈ 40% will be rainy days

5. A taxi company has divided the city into three districts – Westmarket, Oldmarket, and Eastmarket. By keeping track of pickups and deliveries, the company found the following:

   - Of the fares picked up in the Westmarket district, only 10% are dropped off in that district, 50% are taken to the Oldmarket district, and 40% go to the Eastmarket district.
   - Of the fares picked up in the Oldmarket district, 20% are taken to the Westmarket district, 30% stay in the Oldmarket district, and 50% are dropped off in the Eastmarket district.
   - Of the fares picked up in the Eastmarket district, 30% are delivered to each of the Westmarket and Oldmarket districts, while 40% stay in the Eastmarket district.

a. Draw a transition digraph for this Markov Chain.



b. Construct a transition matrix for these data.

$$T = \begin{array}{c} \\ W \\ O \\ E \end{array} \begin{array}{c} \begin{array}{ccc} W & O & E \end{array} \\ \begin{bmatrix} .1 & .5 & .4 \\ .2 & .3 & .5 \\ .3 & .3 & .4 \end{bmatrix} \end{array}$$

c. Write an initial-state matrix for a taxi that starts off by picking up a fare in the Oldmarket district. What is the probability that it will end up in the Oldmarket district after three additional fares?

$$D_0 = \begin{array}{c} \begin{array}{ccc} W & O & E \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$D_0 T^3 = \begin{bmatrix} .22 & .35 & .43 \end{bmatrix}$$

35% chance that the taxi ends up in Oldmarket after 3 additional fares.

Section 2: Game Theory (Part 1)

1. Each of the following matrices represents a payoff matrix for a game. Determine the best strategies for the row and column players. If the game is strictly determined, find the saddle point of the game.

a. $\begin{bmatrix}16 & 8\\ 12 & 4\end{bmatrix}\begin{matrix}=\text{⑧}\\ =4\end{matrix}$ saddle point
$\begin{matrix}\shortparallel & \shortparallel\\ 16 & \text{⑧}\end{matrix}$
yes, s.d.
row 1, col 2

d. $\begin{bmatrix}0 & 1 & 2\\ 3 & -2 & 0\end{bmatrix}\begin{matrix}\text{⓪}\\ -2\end{matrix}$
$\begin{matrix}3 & \text{①} & 2\end{matrix}$
Not strictly determined
Row 1, col. 2

b. $\begin{bmatrix}0 & 4\\ -1 & 2\end{bmatrix}\begin{matrix}=\text{⓪}\\ =-1\end{matrix}$ saddle point
$\begin{matrix}\shortparallel & \shortparallel\\ \text{⓪} & 4\end{matrix}$
yes, s.d.
row 1, col. 1

e. $\begin{bmatrix}0 & -6 & 1\\ -4 & 8 & 2\\ 6 & 5 & 4\end{bmatrix}\begin{matrix}=-6\\ =-4\\ =\text{④}\end{matrix}$ saddle point
$\begin{matrix}\shortparallel & \shortparallel & \\ 6 & 8 & \text{④}\end{matrix}$
yes, s.d.
Row 3, col. 3

c. $\begin{bmatrix}2 & -3\\ -3 & 4\end{bmatrix}\begin{matrix}=-3\\ =\text{③}\end{matrix}$
$\begin{matrix}\shortparallel & \shortparallel\\ \text{②} & 4\end{matrix}$
Not strictly determined

f. $\begin{bmatrix}0 & 3 & 1\\ -3 & 0 & 2\\ -1 & -4 & 0\end{bmatrix}\begin{matrix}=\text{⓪}\\ =-3\\ =-4\end{matrix}$ saddle point
$\begin{matrix}\text{⓪} & 3 & 2\end{matrix}$
yes, s.d.
row 1, col. 1

2. Using the following payoff matrix, answer the questions below: $\begin{bmatrix}-4 & 2\\ 5 & 3\end{bmatrix}\begin{matrix}=-4\\ =\text{③}\end{matrix}$
$\begin{matrix}\shortparallel & \shortparallel\\ 5 & \text{③}\end{matrix}$

a. Determine the best strategies for the row and column players and the saddle point of the game.

Saddle point = 3
Row 2; column 2 (Best Strategies)

b. Add 4 to each element in the payoff matrix. How does this affect the best strategies and the saddle point of the game?

$\begin{bmatrix}0 & 6\\ 9 & 7\end{bmatrix}\begin{matrix}=\text{⓪}\\ \text{⑦}\end{matrix}$
$\begin{matrix}9 & \text{⑦}\end{matrix}$
Adds 4 to saddle point.
Strategies stay same.

c. Multiply each element in the payoff matrix by 2. How does this affect the saddle point of the game and the best strategies?

$\begin{bmatrix}-8 & 4\\ 10 & 6\end{bmatrix}\begin{matrix}=-8\\ =\text{⑥}\end{matrix}$
$\begin{matrix}10 & \text{⑥}\end{matrix}$
Doubles saddle point.
strategies stay same.

d. Make a conjecture based on the results of parts b and c.

If a payoff matrix adds a constant or is mult. by a constant, the saddle point changes respectively, and best strategies remain the same.

3. Two major discount companies, Salemart and Bestdeal, are planning to locate stores in Nebraska.
    - If Salemart locates in City A and Bestdeal in City B, then Salemart can expect an annual profit of $50,000 more than Bestdeal's annual profit.
    - If both locate in City A, they expect equal profits.
    - If Salemart locates in City B and Bestdeal in City A, then Bestdeal's profits will exceed Salemart's by $25,000.
    - If both companies locate in City B, then Salemart's profits will exceed Bestdeal's by $10,000.

    What are the best strategies in this situation and what is the saddle point of the game?

$$\begin{array}{cc|cc|c} & & \multicolumn{2}{c}{\text{Bestdeal}} & \\ & & A & B & \\ \text{Salemart} & A & 0 & 50{,}000 & =\textcircled{0} \\ & B & -25{,}000 & 10{,}000 & = -25{,}000 \\ & & \textcircled{0} & 50{,}000 & \end{array}$$

Saddle point = 0

Best strategy: Both should locate in city A [Equal profits]

4. Mike is going home to see his wife Nancy when he suddenly remembers that today may be a special anniversary. He always brings her a single red rose on this occasion. But he's not sure. Maybe the anniversary is next week. What should he do?
    - If it is their anniversary and he doesn't bring a rose, then he'll be in big trouble. On a scale from 0 to 10, he'd score a -10.
    - If he doesn't bring a rose and it isn't their anniversary, Nancy won't know anything about his frustration and he'll score a 0.
    - If he brings a rose and it is not their anniversary, then Nancy will be suspicious that something funny is going on but he'll score about a 2.
    - If it is their special anniversary and he brings a rose, then Nancy will be expecting it and he'll score a 5.

    Write a payoff matrix for this situation. What is Mike's best strategy?

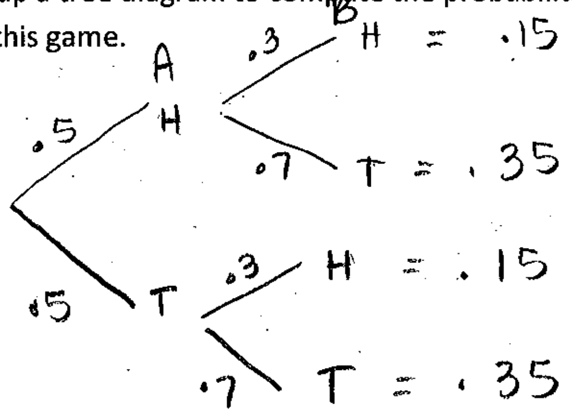
$$\begin{array}{cc|cc|c} & & \multicolumn{2}{c}{\text{Nancy}} & \\ & & \text{Ann.} & \text{Not Ann.} & \\ \text{Mike} & \text{Rose} & 5 & 2 & \textcircled{2} \\ & \text{No Rose} & -10 & 0 & -10 \\ & & 5 & \textcircled{2} & \end{array}$$

Mike should buy a rose. Since the saddle point = 2, the worst he could do is score a 2.

Section 3: Game Theory (Part 2)

1. Suppose that in the example of this lesson, Person A decides to return to flipping his coin while Person B continues to pursue her best strategy of playing heads three-tenths of the time.

   a. Set up a tree diagram to compute the probabilities of each of the four outcomes for this game.



   b. What is the probability that both A and B will show heads? .15

   c. What is the probability that B will show tails and A will show heads? .35

   d. What is the probability that B will show heads and A will show tails? .15

   e. What is the probability that both A and B will show tails? .35

   f. Write a probability distribution chart for A's winnings.

Outcome	HH	HT	TH	TT
Prob	.15	.35	.15	.35
Amt won	4	−2	−3	1

   g. Calculate A's expected payoff for this game. Explain what this means in terms of pennies lost or won.

$$.15(4) + .35(-2) + .15(-3) + .35(1) = -.2$$

Person A loses 2 pennies for every 10 plays.

   h. How does this payoff compare with A's expected payoff if he plays his best strategy as computed in this lesson?

Exactly the same (poor person A!)

2. Suppose that Person A and Person B change their game so that the payoffs to A are:

$$\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$$

a. Use the row matrix $[p \quad 1-p]$ to find A's best strategy for this game.

$$[p \quad 1-p]\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} = [3p + -2(1-p) \qquad -2p + (1-p)]$$

$3p - 2 + 2p = -2p + 1 - p$

$5p - 2 = -3p + 1 \qquad 8p = 3 \qquad p = \frac{3}{8}$

$.375$

$1 - p = \frac{5}{8}$

$.625$

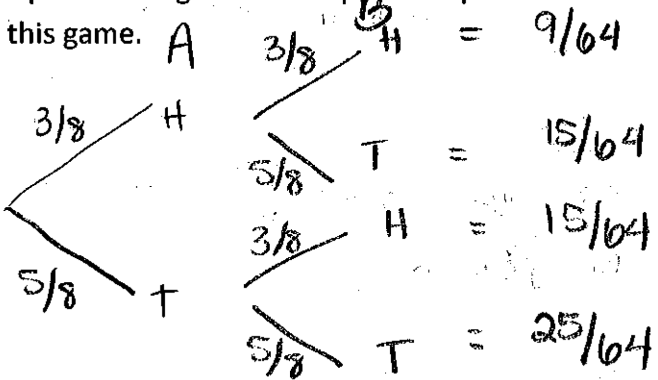
b. Use the column matrix $\begin{bmatrix} q \\ 1-q \end{bmatrix}$ to find B's best strategy for this game.

$$\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} q \\ 1-q \end{bmatrix} = [3q + -2(1-q) \qquad -2q + 1 - q]$$

$5q - 2 = -3q + 1 \qquad q = 3/8 = .375$

$1 - q = 5/8 = .625$

c. Set up a tree diagram to compute the probabilities of each of the four outcomes for this game.



d. Prepare a probability distribution chart for A's winnings.

Outcome	HH	HT	TH	TT
Probability	9/64	15/64	15/64	25/64
Amt. won	3	-2	-2	1

e. Find A's expected payoff for this game.

$9/64(3) + 15/64(-2) + 15/64(-2) + 25/64(1)$

$27/64 - 30/64 - 30/64 + 25/64 = -8/64 = -1/8$

f. Interpret your answer in part *e* in terms of how many pennies A can expect to win or lose over a number of games.

Lose one penny every 8 games

3. In a game known as Two-Finger Morra, two players simultaneously hold up either one or two fingers. If they hold up the same number of fingers, player 1 wins from player 2 the sum (in pennies) of the digits held up on both players' hands. If they hold up different numbers, then player 2 wins the sum from player 1.

   a. Write the payoff matrix for this game.

Player 2

$$\text{Player1} \quad \begin{array}{c|cc} & 1 & 2 \\ 1 & 2 & -3 \\ 2 & -3 & 4 \end{array} \quad \boxed{\begin{matrix} -3 \\ -3 \end{matrix}}$$

   b. Find the best strategy for each player.

② 4

$$[p \quad 1-p] \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix} = [2p + -3(1-p) \quad -3p + 4(1-p)]$$

$$5p - 3 = -7p + 4$$

$$12p = 7$$

$$\begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} q \\ 1-q \end{bmatrix} \rightarrow \text{same}$$

Show 1 finger

$p = 7/12$

$q = 7/12$

   c. Find the expected payoff for the row player.

7/12 1 — 7/12 1 $= 49/144$ (2)

5/12 2 $= 35/144$ (-3)

5/12 2 — 7/12 1 $= 35/144$ (-3)

5/12 2 $= 25/144$ (4)

$= -1/12$ Lose 1¢ for every 12 plays.

   d. Is this a fair game? Explain your answer.

Because the expected payoff is not 0, this is not a fair game.

4. In a campaign for student council president, the top two candidates, Amber and Ben, are making two promises about what they will do if they are elected. The payoff matrix in terms of the number of votes Amber will gain follows. What is the best strategy for each candidate and what is Amber's expected payoff?

$$\begin{array}{cc} & \text{Ben} \\ \text{Amber} & \begin{bmatrix} 200 & 100 \\ 50 & 180 \end{bmatrix} \end{array} \qquad \begin{array}{c} \text{Ben} \\ \begin{bmatrix} q \\ 1-q \end{bmatrix} \end{array}$$

Amber

$[p \quad 1-p] * \nearrow$

Amber

$$= [200p + 50(1-p) \quad 100p + 180(1-p)]$$

$$150p + 50 = -80p + 180$$

$$230p = 130 \quad p = \frac{13}{23} \text{ (Prom. A)}$$

$$\frac{10}{23} \text{ (Prom. B)}$$

Ben

$$200q + 100(1-q) = 50q + 180(1-q)$$

$$100q + 100 = -130q + 180$$

$$230q = 80$$

$$q = 8/23 \text{ (Prom. A)}$$

$$15/23 \text{ (Prom. B)}$$

Payoff!

13/23 — 8/23 $= \frac{104}{529}(200)$

15/23 $= \frac{195}{529}(100)$

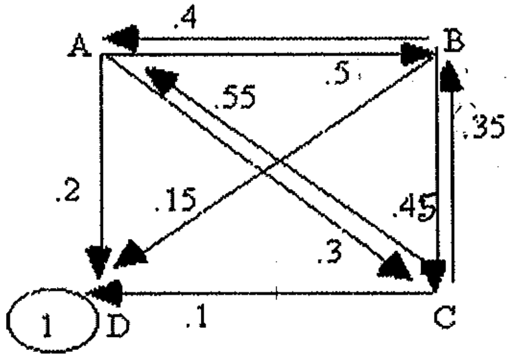
10/23 — 8/23 $= \frac{80}{529}(50)$

15/23 $= \frac{150}{529}(180)$

$= 134.78$ or $\boxed{135 \text{ votes}}$

Chapter 6 Review ☺

1. Given below is a digraph giving the transition probabilities among states A, B, C, and D.



a. Write a transition matrix for the digraph. Label the rows and columns.

$$T = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0.5 & 0.3 & 0.2 \\ 0.4 & 0 & 0.45 & 0.15 \\ 0.55 & 0.35 & 0 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

b. If the initial distribution is state A at 100%, and state B, state C, and state D each at 0%, find the distribution after 10 transitions. After 30 transitions. After 60 transitions.

$$D_0 = \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \cdot T^{10} = \begin{bmatrix} 0.0654 & 0.0619 & 0.0582 & 0.8165 \end{bmatrix}$$

$$\cdot\ T^{30} = \begin{bmatrix} 0.0024 & 0.0023 & 0.0020 & 0.9933 \end{bmatrix}$$

$$\cdot\ T^{60} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

c. If the initial distribution is state A, state B, state C, and state D each at 25%, find the distribution after 10 transitions. After 30 transitions. After 60 transitions.

$$D_0 = \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix} \end{array} \cdot \quad T^{10} = \begin{bmatrix} 0.0513 & 0.0484 & 0.0438 & 0.8565 \end{bmatrix}$$

$$T^{30} = \begin{bmatrix} 0.0019 & 0.0018 & 0.0016 & 0.9948 \end{bmatrix}$$

$$T^{60} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

d. If the initial distribution is state D at 100%, and state A, state B, and state C each at 0%, find the distribution after 10 transitions. After 30 transitions. After 60 transitions.

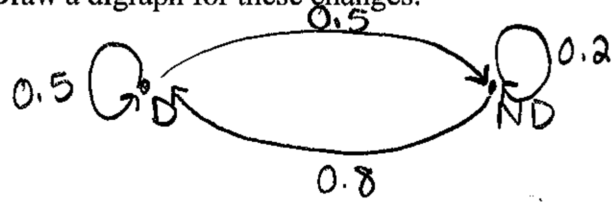
$$D_0 = \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \cdot \quad T^{10} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T^{30} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T^{60} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

2. Doug and Kaye date on an irregular basis. If they went out last weekend, there is a 50% chance they will date the next weekend. If there was no date last weekend, there is an 80% chance that they will go out the next weekend.

a. Draw a digraph for these changes.



b. Set up a transition matrix for the changes. Label the rows and columns.

$$T = \begin{array}{c|cc} & D & ND \\ \hline D & 0.5 & 0.5 \\ ND & 0.8 & 0.2 \end{array}$$

c. If Doug and Kaye dated last weekend, how likely is it that they will have a date <u>four weeks</u> from then?

$$D_0 = \begin{array}{cc} D & ND \\ \end{array} \left[\begin{array}{cc} 1 & 0 \end{array}\right] \cdot T^4 = \begin{array}{cc} D & N.D \end{array} \left[\begin{array}{cc} .6185 & .3815 \end{array}\right]$$

61.85% chance

d. If this trend continues, how likely are Doug and Kaye to date?

$$D_0 T^{10} \longrightarrow 61.54\%$$

3. Each of the following matrices represents a payoff matrix for a game. If the game is strictly determined, find the saddle point of the game. If the game is not strictly determined, explain why not.

a. $$\begin{array}{cc} \left[\begin{array}{cc} -2 & 5 \\ 2 & 3 \end{array}\right] & \begin{array}{c} -2 \\ \textcircled{2} \end{array} \\ \begin{array}{cc} \textcircled{2} & 5 \end{array} & \end{array}$$

saddle point = 2
strictly determined

b. $$\begin{array}{cc} \left[\begin{array}{cc} -3 & 1 \\ 2 & -2 \end{array}\right] & \begin{array}{c} -3 \\ \textcircled{-2} \end{array} \\ \begin{array}{cc} 2 & \textcircled{1} \end{array} & \end{array}$$

no saddle point
not strictly determined
maximin = -2
minimax = 1

c. $$\begin{array}{cc} \left[\begin{array}{cc} 0 & -4 \\ 1 & -2 \end{array}\right] & \begin{array}{c} -4 \\ \textcircled{-2} \end{array} \\ \begin{array}{cc} 1 & \textcircled{-2} \end{array} & \end{array}$$

saddle point = -2
strictly determined

4. The matrix below represents a payoff matrix for a game. In playing the game the row player plays row one .4 of the time and row two .6 of the time. If the column player always plays column one, what is the expected payoff of the game for the row player?

$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & -2 & 3 \\ 2 & 1 & 2 \end{array}$$

1, 1	1, 2	2, 1	2,2
-2	3	1	2
.4	0	.6	0

$$-2(.4) + 3(0) + 1(.6) + 2(0) = \boxed{-.2}$$

5. The matrix below represents a payoff matrix for a game.

$$\begin{bmatrix} 6 & -2 \\ -1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} p & 1-p \end{bmatrix} \cdot \begin{bmatrix} 6 & -2 \\ -1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 6p - 1(1-p) = -2p + 4(1-p) \end{bmatrix}$$

$$6p - 1 + p = -2p + 4 - 4p$$

$$7p - 1 = -6p + 4$$

$$13p = 5 \qquad p = 5/13$$

a. Calculate the best strategy for the row player.

Play row 1 5/13 of the time
row 2 8/13 of the time

b. Calculate the best strategy for the column player.

$$\begin{bmatrix} 6 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} q \\ 1-q \end{bmatrix} = \begin{bmatrix} 6q - 2(1-q) \\ -1q + 4(1-q) \end{bmatrix}$$

Play col 1 6/13 of the time
col 2 7/13 of the time

$$6q - 2 + 2q = -1q + 4 - 4q$$

$$8q - 2 = -5q + 4$$

$$13q = 6 \qquad q = 6/13$$

c. If both players play their best strategy what is the expected payoff for the row player?

+1.69

6. The Boar and the Rooster each want to be elected King of the Barnyard. The Boar has two strategies—he can display his teeth or throw his weight around. The Rooster, however, has three strategies—he can crow loudly, strut through the barnyard or fly up into the trees. If the Boar displays his teeth, he will gain two votes if the Rooster crows, lose four votes if the Rooster struts around or gain one vote if the Rooster flies into the treetops. If the Boar throws his weight around, he will gain six votes if the Rooster crows, lose three votes if the Rooster struts around the barnyard, and gain two votes if the Rooster flies into a tree. What is the best strategy for each contestant? Answer in terms of game theory.

$$\begin{array}{cc|ccc|c} & & & \text{Rooster} & & \\ & & \text{crow} & \text{strut} & \text{Fly} & \\ \hline \text{Boar} & \text{Teeth} & 2 & -4 & 1 & -4 \\ & \text{weight} & 6 & -3 & 2 & \textcircled{-3} \\ \hline & & 6 & \textcircled{-3} & 2 & \end{array}$$

strictly determined w/ Saddle point -3

The Boar should throw his weight around and the Rooster should strut through the barnyard.