Unit 3 Notes: Graphs and Circuits

Section 3.1: Modeling with Graphs

Konigsberg Bridge Problem: "Is it possible for a person to walk around the city crossing each bridge exactly once, starting and ending at the same point?"

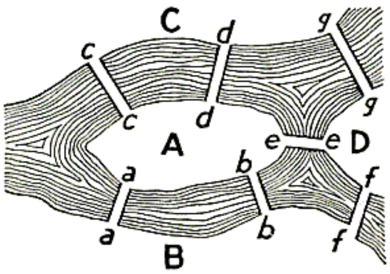
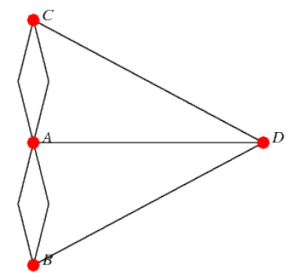


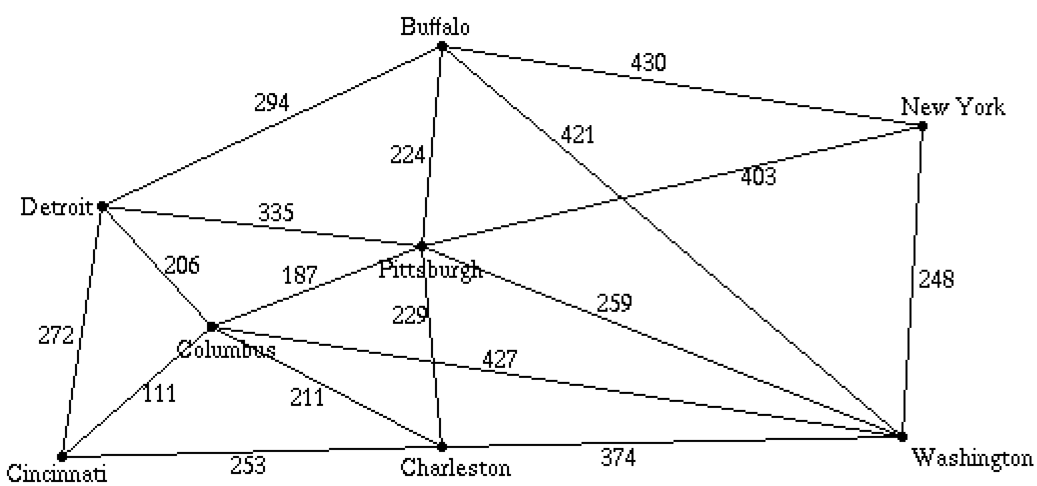
FIGURE 98. *Geographic Map: The Königsberg Bridges.*



Euler constructed the simplified geometric model of the situation called a graph, thus creating the subject of **graph theory**. These types of graphs are not the same as graphs of functions or relations (like in Algebra). Graphs are used to solve a wide variety of problems.

ACTIVITY: What's the Shortest Route?

Suppose a salesperson wishes to travel to each city in the map below exactly once, starting and ending in New York, and using only the roads shown. The numbers on the roads indicate distances (in miles) between cities. Find the shortest route that the salesperson could use.



Work in small groups to come up with your route and the justification for why you chose the route you did. Look for a method that will work every time you try a problem like this. Are you convinced that your route is the shortest? Why?

Example 1:

Building a house is usually a team effort that involves specialists, such as architects, excavators, concrete workers, framing carpenters, roofers, etc. Different specialists are often able to work at the same time provided that the work that must precede a particular specialist is completed before that specialist begins. By working simultaneously whenever possible, the house can be completed more quickly, and it is natural to wonder if there is an optimal way to schedule the various tasks for completion.

Information to build a house:

	Task	**Time (Days)**	**Immediately Preceding Task**
A	Preparing final house and site plans	3	None
B	Excavation and foundation construction	5	A
C	Framing and closing main structure	12	B
D	Plumbing	5	C
E	Wiring	3	C
F	Heating-cooling installation	7	E
G	Insulation and dry wall	9	D, F
H	Exterior siding, trim, and painting	15	C
I	Interior finishing and painting	7	G
J	Carpeting	3	I
K	Landscaping	4	H

If we did each task individually, it would take 73 working days to complete the house. Draw a graph we could use to help the builder decide which tasks can be done simultaneously in order to complete the job more quickly. Then use the graph to determine the least number of days to complete the house.

Notice that the algorithm used in the example above is recursive (depends on the previous term). Here is the general algorithm to calculate the number of days for a particular task:

If there are no prerequisite tasks, use the number of days required by the task alone.

Otherwise:

(1) Calculate the number of days for each prerequisite task by using the algorithm.
(2) Choose the largest of the numbers found in step 1, and add to it the number of days required by this task alone.

Definition: When you can travel along each edge of a graph in only one direction, the graph is

called a ____________________________________ or _______________________.

(A sample of such a graph is a probability tree!)

Example 2:

60% of the students in a college live on campus. 70% of those who live on campus favor a tuition increase to pay for improved student health services. 40% of those who live off campus favor this increase. Draw a graph to determine the following:

(a) What is the proportion of students who favor this increase to pay for improved student health services?
(b) If a randomly selected student favors the tuition increase, what is the probability that the student lives on campus?

Section 3.2: The Definition of a Graph

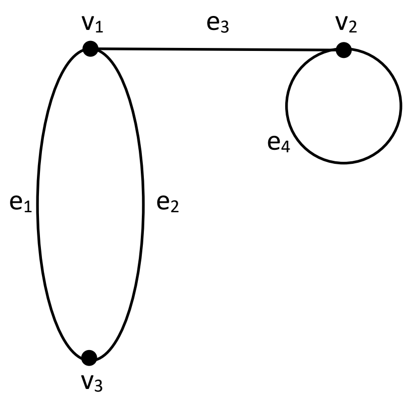
A **graph** consists of:

1. A finite set of ____________________,

2. A finite set of ____________________,

3. A function (called the _______________________ function) that maps each edge to a set of either one or two vertices (the _______________ of the edge).

Consider the graph and table below:



edge	endpoints
e_1	$\{v_1, v_3\}$
e_2	$\{v_1, v_3\}$
e_3	$\{v_1, v_2\}$
e_4	$\{v_2\}$

The vertices are: ______________________ The edges are: ______________________

Definition: This table/graph represents an ____________________________ function.

The essential feature of an edge is that it ________________ its endpoints. The SHAPE (or curve) is not important!

Definition: Two vertices connected by an edge are _________________________.

Definition: Two edges with a common endpoint are called _________________________.

CAUTION: Edges in graphs (unlike edges in shapes/polygons) have no points other than their endpoints. Just because two edges "cross" does not mean there is a vertex!

Note that v_1 and v_3 are connected by more than one edge. When this occurs then we say that the edges are ___________________. Also, edge e_4 joins vertex v_2 to itself. Such an edge is called a _______________. In all graphs, there MUST be a vertex at each end of each edge.

Example 1:

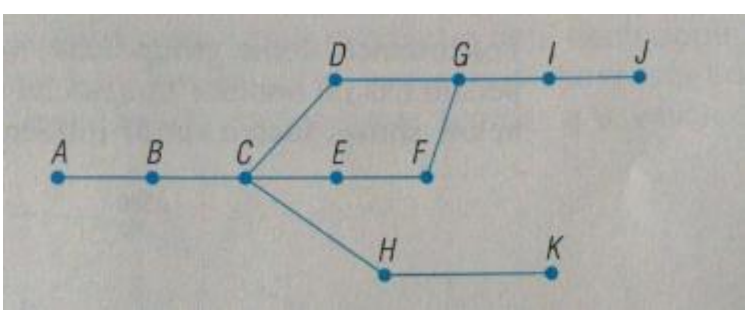
Draw a picture of the graph *G* defined as follows.

1. Set of vertices: $\{v_1, v_2, v_3, v_4, v_5\}$
2. Set of edges: $\{e_1, e_2, e_3, e_4, e_5\}$
3. Edge-endpoint function:

Edge	Endpoints
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_4\}$
e_3	$\{v_1, v_4\}$
e_4	$\{v_5\}$
e_5	$\{v_4 v_5\}$

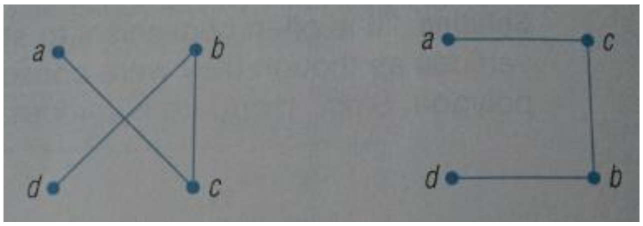
Notice that v_3 is not the endpoint of any edge. That vertex is called an _____________________. Although all edges must have endpoints, a vertex need not be the endpoint of an edge.

Take a look at the house-building problem from 4.1:



In this graph, there are no loops or parallel edges. Such a graph is called ___________________.

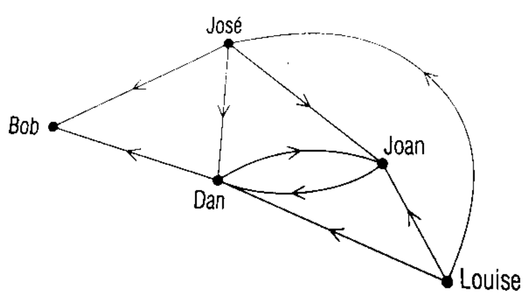
Consider the simple graph with vertices *a, b, c, d* and edges {*a, c*}, {*b, d*}, and {*b, c*}. Two pictures of this graph are shown below. The picture on the left has two edges that cross, but note that they do NOT intersect (there is not a vertex at the point of intersection). This type of intersection is called a ___________________. The figure on the right illustrates the same graph, but it avoids all crossings.



<u>Example 2:</u> Draw all simple graphs with vertices {*u, v, w*} if one of the edges is {*u, v*}.

Sometimes it is useful to add direction to each edge of a graph. We call this kind of graph a **digraph**. Instead of writing $\{v_1, v_2\}$ for the endpoints of an edge, in a digraph we would write the endpoints as an ordered pair (v_1, v_2).

For instance, some group-behavior studies investigate the influence one person has on another in a social setting. The directed graph pictured below shows such a set of influence relationships. (NOTE: This is the same thing as the communication networks we learned about in Unit 2!)



Who does Jose influence?

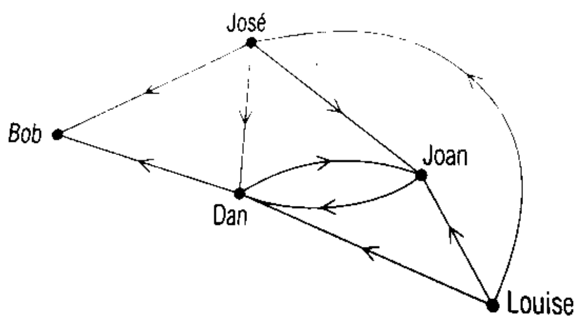
Who does Louise influence?

Surprisingly, we can describe a graph numerically, even when no numbers are given. We can do this using a matrix! (YEAH!)

Definition: The **adjacency matrix** *M* for a graph with vertices $v_1, v_2, \ldots, v_n$ is the *n* x *n* matrix in which the element in the *i*th row and *j*th column is the number of edges from vertex v_i to vertex v_j.

Example 3:

Write the adjacency matrix for the directed graph of influence relationships pictured below.



Example 4:

Draw a picture of a graph (not directed) that has the following adjacency matrix.

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Section 3.3: Handshake Problems

In groups of 6, work through the following problems:

In sub-groups of the size indicated, everyone shakes hands with everyone else. Record your data in the table below:

Number of People in Group	Total Number of Handshakes
1	
2	
3	
4	
5	
6	

Represent these sets of data with graphs. Label your graphs H_n, where *n* is the number of people in a group.

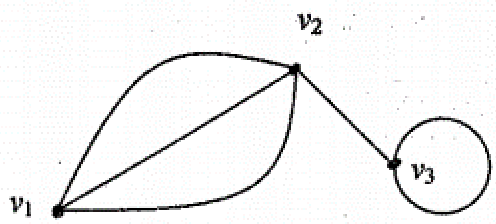
H_1 H_2 H_3

H_4 H_5 H_6

H_n is called a **complete graph** because every vertex is connected to every other vertex by an edge. Since every person shakes hands once with each other person, every pair of vertices is joined by exactly one edge.

Definitions: If v is a vertex of a graph G, the **degree of** v, denoted $\mathbf{deg(v)}$, equals the number of edges that have v as an endpoint, with each edge that is a loop counted twice. The **total degree of** G is the sum of the degrees of all the vertices of G.

Example - Given the graph, G

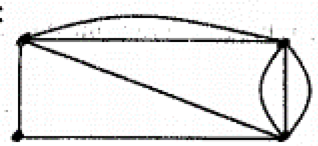


$deg(v_1) = 3$
$deg(v_2) =$ ____
$deg(v_3) =$ ____

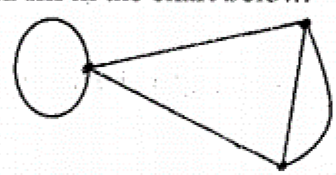
Total degree of $G =$ ______

Now examine the following graphs and fill in the chart below.

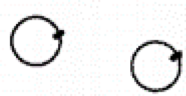
Q:



R:



S:



T:

Graph	Number of edges	Total Degree
Q		
R		
S		
T		
H_2		
H_3		

Can you find a relationship between the number of edges in a graph and the total degree?

Why do you think this relationship exists?

What does this relationship tell you about the total degree of any graph?

Now pick 5 people in your group. Have each of the 5 people shake hands with exactly 3 others in the group of 5. What happens?

Try to draw a graph of this situation.

Now try to have the group of 5 shake hands with exactly 1, 2, and 4 others. When does this experiment fail? When is it successful?

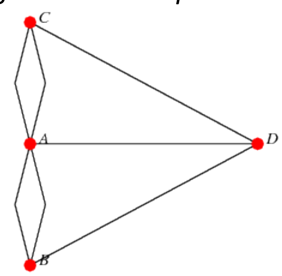
How about a different sized group? Try some others with varying numbers of handshakes. Record or graph your results.

Can you observe any patterns or make any generalizations about the experiments you just tried?

Extension Problem: Can you find a formula for the total number of handshakes for a group of *n* people?

Section 3.4: The Konigsberg Bridge Problem

The beginning of this unit posed the Konigsberg Bridge Problem: "*Is it possible for a person to walk around the city crossing each bridge exactly once, starting and ending at the same point?*"



Euler's solution needs some vocab to help us out:

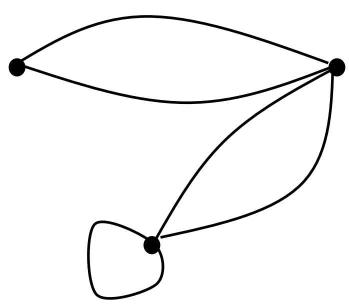
Definitions:

Suppose G is a graph and *v* and *w* are vertices of G.

1. A ___________________ from ______________ is an alternating sequence of adjacent vertices and edges beginning with ________ and ending with __________.
2. A ___________________ from _______________ is a **walk** from _______ to ________ in which no edge is repeated.
3. A __________________ is a **path** that starts and ends at the same vertex.
4. An __________________________ is a **circuit** that contains every edge and vertex of G.

Using this language, we can no re-state the problem: "*Does the Konigsberg Bridge problem have an Euler Circuit?*"

Use this example to help you out. (Label all of the vertices and edges.)



Name a walk from v_1 to v_3. Is that walk a path? Why or why not?

Name a circuit from v_1 to v_1.

Is there an Euler Circuit in this graph? If so, find it. If not, explain why.

Fill in the table below to help show the differences among walks, paths, circuits, and Euler circuits (Fill in each box with “yes”, “no”, or “allowed”)

	Repeated edge?	Start/End same point?	Includes all vertices/edges?
Walk			
Path			
Circuit			
Euler Circuit			

Euler Circuit Theorem:

If a graph has an Euler circuit, then every ________________ of the graph has an _________ degree.

Knowing the definition above, you can now solve the Konigsberg Bridge Problem…. Will it work?

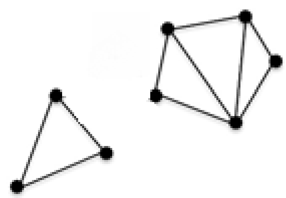
Does the converse of the Euler Circuit Theorem work? Is it true that if every vertex has an even degree, then the graph has an Euler Circuit? If not, show a graph that wouldn’t work.

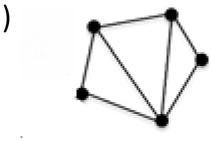
This leads to the concept of ‘connectedness’.

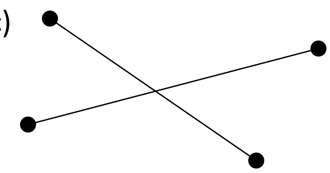
Definition: Suppose G is a graph. Two vertices *v* and *w* in G are _______________ vertices if and only if there is a ________________ in G from *v* to *w*.

Example 1:

Tell whether or not each graph is connected:

a) 

b) 

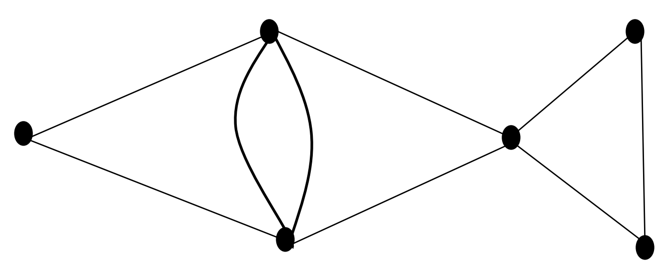
c) 

With the idea of connectedness, we can add on to the Euler circuit theorem:

Theorem: If a graph G is connected and every vertex of G has ___________ degree, then G has an Euler circuit.

Example 2:

Does the following graph have an Euler circuit? If so, find such a circuit.



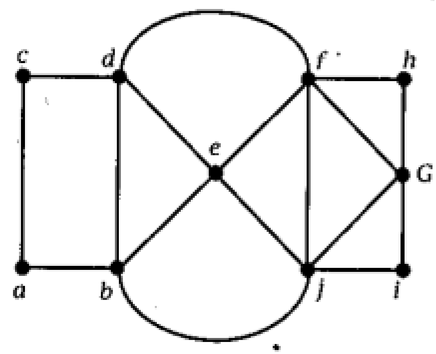
Theorem:

If a ______________________ graph contains a ________________ and an edge is removed

from the circuit, then the resulting graph is also ______________________.

Section 3.5: Hamiltonian Circuits and Graph Coloring

Suppose that you are a city inspector, but instead of inspecting all of the streets in an efficient manner, you must inspect the fire hydrants that are located at every intersection. This implies that you are searching for an optimal route that begins at garage, G, visits each intersection exactly once, and returns to the garage.



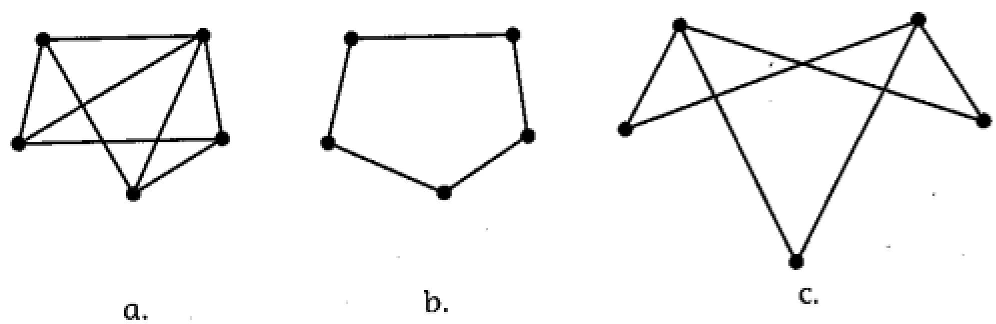
Find a path that meets this criteria:____.

A path that uses each vertex of a graph exactly once is known as a _______________________.

If the path ends at the starting vertex, it is called a ____________________________________.

Example 1:

Find a Hamiltonian circuit for each graph below (if one exists):



As with Euler circuits, often it is useful for the edges of the graph to have direction. Consider a competition in which each player must play every other player. By using directed edges, it is possible to indicate winners and losers. To illustrate this, draw a complete graph in which the vertices represent the players, and a directed edge from A to B indicates that A defeats player B. This type of graph is known as a **tournament**.

Every tournament contains at least one ______________________________. If there is exactly one, it can be used to rank the teams in order, from winner to loser.

Example 2:

Suppose four teams play in the school soccer round-robin tournament. The results of the competition follow:

Game	AB	AC	AD	BC	BD	CD
Winner	B	A	D	B	D	D

Draw a graph to represent the tournament. Find a Hamiltonian path and use it to rank the participants from winner to loser.

Graph Coloring

When it's time to schedule meetings or register for new classes, scheduling conflicts often arise. Mathematicians have found that graphs are useful tools in helping to resolve these conflicts.

Explore This: Below is a table of clubs at Central High School and students who hold offices in these clubs:

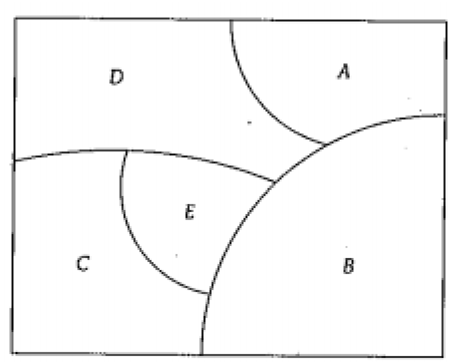
	Math Club	Chess Club	Science Club	Art Club	Pep Club	Spanish Club
Nicole	x	x	x			
Maddie	x			x	x	
Quinn		x				x
Emma	x		x			
Alex	x				x	

Each club wants to meet once a week. Since several students hold offices in more than one organization, it is necessary to arrange the meeting days so that no students are scheduled for more than one meeting on the same day. Is it possible to create such a schedule? What is the minimum number of days needed (we want the fewest number of days possible)?

Problems of this type are called **coloring** problems because historically the labels placed on the vertices of the graphs were referred to as *colors*. The process of labeling the graph is called **coloring the graph**. The minimum number of labels that can be used is known as the **chromatic number** of the graph. What is the chromatic number for the graph above?

<u>Example 3:</u>

Use the concept of coloring to color the following map using four or fewer colors.

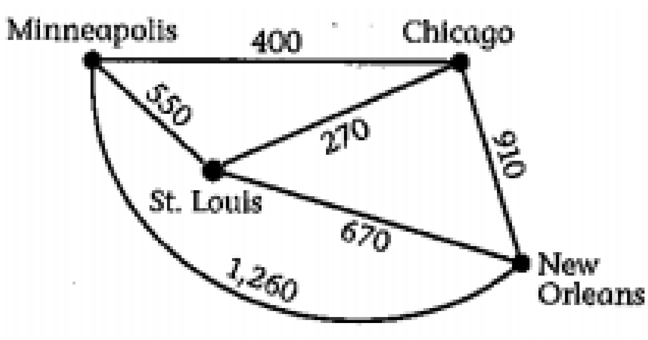


1. Represent the map with a graph where each vertex represents a region on the map.
2. Draw edges between vertices if the regions on the map have a common border.
3. Label the graph using a minimum number of colors.

Section 3.6: The Traveling Salesperson Problem/Finding the Shortest Route

A **traveling salesperson problem (TSP)** involves finding a Hamiltonian circuit or minimum value such as time, distance or cost. Optimization problems of this type are becoming increasingly important in the world of communications, warehousing (AMAZON!), airline networking, delivery truck routing, and building wiring.

Example 1: Suppose you are a salesperson who lives in St. Louis. Once a week, you have to travel to Minneapolis, Chicago, and New Orleans and then return home. The graph below represents the trips that are available to you. The edges represent cost. When each edge is assigned a number, we call it a **weighted graph**.



To save money, we want to find the least expensive route that begins in St. Louis, visits each of the other cities exactly once, and returns home. One way to solve this problem is the **brute force method**, by listing every possible circuit. Draw a tree diagram to list each possible circuit.

Doing a tree diagram isn't always the most efficient method, but it will find us the optimal path!

What if we didn't need to visit every vertex and return to the starting point, but instead you only needed to find the shortest path from one vertex in the graph to another?

Shortest Path Algorithm

1. Label the starting vertex S and circle it. Examine all edges that have S as an endpoint. Darken/highlight the edge with the shortest length and circle the vertex at the other endpoint of the darkened edge.
2. Examine all uncircled vertices that are adjacent to the circled vertices.
3. Using only circled vertices and darkened edges between the vertices circled, find the lengths of all paths from S to each vertex being examined. Choose the vertex and the edge that yield the shortest path. Circle this vertex and darken this edge. Ties are broken arbitrarily.
4. Repeat steps 2 and 3 until all vertices are circled. The darkened edges of the graph form the shortest routes from S to every other vertex in the graph.

<u>Example 2:</u> Use the shortest path algorithm to find the shortest path from A to F in the graph.