<u>Section 1: Addition and Subtraction of Matrices</u>

Learning Targets:

- I can use matrix terminology and definitions.
- I can find the sum and difference of two matrices.
- I can use matrices to model real-world situations.

Suppose you and your friends want to plan a pizza party. You decide to call a few pizza places around Holland to ask about prices for large single-topping pizzas, a 2 liter of Coke, and breadsticks. You record the information in a table:

	Sluggo's	Little Caesar's	Papa John's	Hungry Howie's
Pizza	$8.99	$5.55	$9.49	$7.95
Drinks	$1.49	$1.99	$1.99	$1.79
Breadsticks	$3.99	$2.95	$5.99	$4.99

Because you are all math nerds, you decide that instead of a table, you would rather use a matrix to represent the data instead:

$$A = \begin{bmatrix} \underline{\hspace{1.5em}} & \underline{\hspace{1.5em}} & \underline{\hspace{1.5em}} & \underline{\hspace{1.5em}} \\ \underline{\hspace{1.5em}} & \underline{\hspace{1.5em}} & \underline{\hspace{1.5em}} & \underline{\hspace{1.5em}} \\ \underline{\hspace{1.5em}} & \underline{\hspace{1.5em}} & \underline{\hspace{1.5em}} & \underline{\hspace{1.5em}} \end{bmatrix}$$

The **dimension** of this matrix is ______ x _____ , meaning it has _____ rows and ____ columns. Each individual entry has a unique name. For example, the amount listed in row 2, column 3 (Papa John's drinks) would be identified as _________.

After looking at your data, you decide to drop Papa John's from the options because they are just too expensive. When you do this, you are left with a matrix with dimensions ____ x ____. We call this a **square matrix** because the number of rows matches the number of columns.

$$\begin{bmatrix} \underline{\hspace{1.5em}} & \underline{\hspace{1.5em}} & \underline{\hspace{1.5em}} \\ \underline{\hspace{1.5em}} & \underline{\hspace{1.5em}} & \underline{\hspace{1.5em}} \\ \underline{\hspace{1.5em}} & \underline{\hspace{1.5em}} & \underline{\hspace{1.5em}} \end{bmatrix}$$

If we are looking at only one restaurant at a time, we would call it a **column matrix**. If we are looking at one item at a time, we would be looking at a **row matrix.**

Matrix Addition/Subtraction

Example 1:

Find A + B, if A = $\begin{bmatrix} -2 & 0 & 1 \\ 0 & 5 & -8 \end{bmatrix}$ and B = $\begin{bmatrix} -6 & 7 & -1 \\ 4 & -3 & -8 \end{bmatrix}$

What would happen if we did B + A? Any difference?

Example 2:

Find C – D, if C = $\begin{bmatrix} 9 & 4 \\ -1 & 3 \\ 0 & -4 \end{bmatrix}$ and D = $\begin{bmatrix} 8 & -2 \\ -6 & 1 \\ 5 & -5 \end{bmatrix}$

What would happen if we did D – C? Any difference?

Example 3:

If $\begin{bmatrix} 2 & 3x & 0 \\ 1 & 5 & z+18 \\ 4 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 5 & -3 \\ 4 & -11y-4 & 8 \end{bmatrix}$, find *x*, *y*, and *z*.

Example 4:

$$\begin{bmatrix} 3x \\ y \end{bmatrix} = \begin{bmatrix} 28 + 4y \\ -3x - 2 \end{bmatrix}$$

Section 2: Multiplying Matrices

Learning Targets:

- I can find scalar multiples of matrices.
- I can find the product of two matrices.
- I can use matrices to model real-world situations.

Definition: Scalar Multiples of a Matrix

When multiplying a matrix by a scalar, you need to multiply _________ entries by the scalar.

Example 1:

If A = $\begin{bmatrix} -4 & 1 & -1 \\ 3 & 7 & 0 \\ -3 & -1 & 8 \end{bmatrix}$, find 3A.

Multiplying matrices

1. Check that the dimensions work:

2. Determine the dimensions of the resulting matrix:

Multiply Matrices You can multiply two matrices if and only if the number of columns in the first matrix is equal to the number of rows in the second matrix.

Multiplication of Matrices	$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} a_1x_1 + b_1x_2 & a_1y_1 + b_1y_2 \\ a_2x_1 + b_2x_2 & a_2y_1 + b_2y_2 \end{bmatrix}$

Multiplicative Properties The Commutative Property of Multiplication does *not* hold for matrices.

Properties of Matrix Multiplication	For any matrices A, B, and C for which the matrix product is defined, and any scalar c, the following properties are true.
Associative Property of Matrix Multiplication	$(AB)C = A(BC)$
Associative Property of Scalar Multiplication	$c(AB) = (cA)B = A(cB)$
Left Distributive Property	$C(A + B) = CA + CB$
Right Distributive Property	$(A + B)C = AC + BC$

Example 2:

Use matrices E = $\begin{bmatrix} 4 & -1 & 2 \\ 0 & 1 & 0 \\ 3 & -2 & 4 \end{bmatrix}$, F = $\begin{bmatrix} 4 & 2 \\ -2 & 3 \end{bmatrix}$, and G = $\begin{bmatrix} 1 & 2 & -3 \\ 3 & 1 & 0 \end{bmatrix}$

a. Find EF

b. Find FG

c. Are there any other multiplication combinations possible? If so, find them.

d. Is matrix multiplication **commutative**?

Example 3:

$A = \begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 5 & -3 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & -2 \\ 6 & 3 \end{bmatrix}$

Find $(A + B)C$

Section 3: Transpose and Inverse Matrices

Learning Targets:

- I can find transpose matrices.
- I can find the product of two matrices to determine if they are inverses.
- I can find the inverse of a matrix.

Definition: The **transpose** (A^T) of a matrix A is the matrix obtained by interchanging the rows and columns of matrix A.

Example 1:

If matrix E = $\begin{bmatrix} 4 & -1 & 2 \\ 0 & 1 & 0 \\ 3 & -2 & 4 \end{bmatrix}$, find E^T.

Identity and Inverse Matrices

The identity matrix for matrix multiplication is a square matrix with 1s for every element of the main diagonal and zeros elsewhere.

Identity Matrix for Multiplication	If A is an $n \times n$ matrix and I is the identity matrix, then $A \cdot I = A$ and $I \cdot A = A$.

If an $n \times n$ matrix A has an inverse A^{-1}, then $A \cdot A^{-1} = A^{-1} \cdot A = I$.

Identity Matrix:

Example 2: Determine whether X and Y are inverse matrices. Find $X \cdot Y$ and $Y \cdot X$

$$X = \begin{bmatrix} 7 & 4 \\ 10 & 6 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 3 & -2 \\ -5 & \frac{7}{2} \end{bmatrix}$$

Example 3:

If matrix E = $\begin{bmatrix} 4 & -1 & 2 \\ 0 & 1 & 0 \\ 3 & -2 & 4 \end{bmatrix}$ and the identity matrix I = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, find IE and EI. What do you notice?

Definition: If A and B are two square matrices such that AB = BA = Identity matrix, then A and B are called inverses of each other (denoted A^{-1}).

Example 4:

Determine if the following matrices are inverses:

A = $\begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$ B = $\begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}$

Find Inverse Matrices

Inverse of a 2 × 2 Matrix	The inverse of a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $ad - bc \neq 0$.

If $ad - bc = 0$, the matrix does not have an inverse.

Example 5:

Find the inverse of $N = \begin{bmatrix} 7 & 2 \\ 2 & 1 \end{bmatrix}$.

First find the value of the determinant.

$$N^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} =$$

Example 6: You Try It!

Find the inverse of each matrix, if it exists.

a. $\begin{bmatrix} 24 & 12 \\ 8 & 4 \end{bmatrix}$

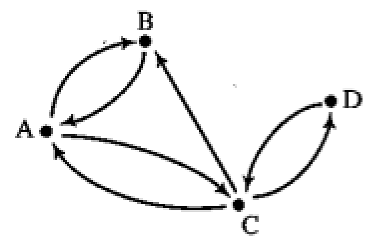
b. $\begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix}$

Section 4: Communication Matrices and Networks

Learning Targets:

- I can solve communication network problems using matrices.

The points A, B, C, and D in the diagram at the right represent four ships at sea that are within communication range. The arrows indicate the direction of radio transmissions. For example, Ship B can send a message to Ship A, but it cannot send a message to Ship C. Although Ship D cannot communicate directly with Ship A, it can communicate indirectly by using Ship C as a relay.



Represent this network as a communication matrix, M:

M =

What do the 0’s on the diagonals represent? Why are they all 0?

While M gives the direct communication links between ships, M^2 would give us the indirect two-step paths that use one ship as a relay.

M^2 =

Example 1:

The matrix M below models the communication between computers P, Q, R, S, and T.

$$\begin{array}{cc} & \text{to} \\ \text{from} & \begin{array}{c|ccccc} & P & Q & R & S & T \\ \hline P & 0 & 1 & 0 & 1 & 1 \\ Q & 1 & 0 & 1 & 1 & 0 \\ R & 0 & 1 & 0 & 0 & 1 \\ S & 1 & 0 & 0 & 0 & 1 \\ T & 1 & 0 & 0 & 0 & 0 \end{array} \end{array} = M$$

A) Draw a diagram that illustrates this communication network.

B) Name the computer(s) that can send data along the greatest number of routes via one relay (two steps).

Example 2:

Matrix M below describes a communication network for weather stations.

$$\begin{array}{cc} & \text{to station} \\ \text{from station} & \begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0 & 0 & 1 & 1 \\ B & 0 & 0 & 0 & 1 \\ C & 1 & 1 & 0 & 1 \\ D & 1 & 0 & 1 & 0 \end{array} \end{array} = M$$

A) To which stations can A send messages?

B) From which stations can B receive messages?

C) Draw a diagram that illustrates this communication network.

D) What information does M^2 give you? (Do not calculate M^2).

E) Will the main diagonal of M^2 necessarily be all zeros?

Section 5: Transition Matrices

Learning Targets:

- I can make predictions using powers of matrices.

Manufacturers do market analysis to predict buying trends of customers in the marketplace over the long run. Their research tells them two things:

(1) Current market shares
(2) Percentages of people who change from brand to brand each month.

The percentages of people who change brands each month can be written as a ***transition matrix***.

Example 1: Of all people who buy Highlight shampoo one month, 90% will buy Highlight the next month, 5% will change to Silky Shine the next month, and 5% will change to another brand. Of those who buy Silky Shine one month, 10% will change to Highlight the next month, 70% will stay with Silky Shine, and 20% will change to another brand. Of the users of other brands, 20% will change to Highlight, and 20% will change to Silky Shine. Write this information as a transition matrix, T.

Example 2: Suppose the market survey also indicates that the current market shares are as follows:

Highlight users, 40%; Silky Shine users, 30%; Other brands, 30%.

We can write this information as a current market share matrix M_0.

Use the transition matrix *T* and the current market share matrix M_0 to calculate the market shares for the next month.

What would you do to find the market shares two months from now?

This is a sample of a **Markov Chain**.

Section 6: Population Growth: The Leslie Model – Part 1

Learning Target:

- I can use matrices (the Leslie model) to determine the growth of a population.

Background (NERD ALERT): When we know the age distribution of a population at a certain date and the birth and survival rates for age-specific groups, you can use this data to create a mathematical model. You can use your model to determine the age distributions of the survivors and descendants of the original population at successive intervals of time. These calculations come in handy for urban planners (knowing how many people there will be in various age groups after certain periods of time have passed) and wildlife managers (keeping animal populations at levels that can be supported in their natural habitats).

P.H. Leslie created an imaginary species to model his growth rate of a population method: small brown rats.

Some assumptions Leslie makes in his model:

1. Only females.

2. Birth rate and survival rates are constant.

3. Survival = probability of survival and moves up age groups

4. Lifespan is only 15 – 18 months

5. Rats have their first litter @ 3 months old and continue every 3 months until age 15 months.

Birth rates and age-specific survival rates for 3-month periods are summarized below:

Age (Months)	Birth Rate	Survival Rate
0-3	0	0.6
3-6	0.3	0.9
6-9	0.8	0.9
9-12	0.7	0.8
12-15	0.4	0.6
15-18	0	0

The initial rat population distribution can be summarized below:

Age (Months	0-3	3-6	6-9	9-12	12-15	15-18
Number	15	9	13	5	0	0

Leslie found that the number of new births after 3 months (1 cycle) was found by ___________

the number of females in the age group by the ____________________. Then, you had to find the ___________.

What was the total number of female rats in the 0 – 3 age group ("new births") after 3 months?

The number of female rats who survive in each age group and move up are:

Age	**Number**	**Survival Rate**	**# Moving up**
0-3	15	0.6	
3-6	9	0.9	
6-9	13	0.9	
9-12	5	0.8	
12-15	0	0.6	
15-18	0	0	

What is the new distribution after 3 months (1 cycle)?

Age (months)	0 – 3	3 – 6	6 – 9	9 – 12	12 – 15	15 – 18
Number						

After all of the populations move up to the next age group, notice that the numbers are not rounded to the nearest integer value. Why is that?

Section 7: Population Growth: The Leslie Model – Part 2

Learning Targets:

- I can calculate population growth using a Leslie Matrix.

As you noticed at the end of the last section, doing the arithmetic 4 cycles out can be quite overwhelming. Using the Leslie model, we can simplify this process down:

Take your original population distribution (P_0) times a matrix that we will call *L*. This will help us calculate the population distribution at the end of cycle 1 (P_1).

$$P_0 L = \begin{bmatrix} 15 & 9 & 13 & 5 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0.6 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0.9 & 0 & 0 & 0 \\ 0.8 & 0 & 0 & 0.9 & 0 & 0 \\ 0.7 & 0 & 0 & 0 & 0.8 & 0 \\ 0.4 & 0 & 0 & 0 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} =$$

Matrix *L* is called the **Leslie Matrix**. This matrix joins the column matrix containing the birth rates of each age group and a series of column matrices that contain the survival rates.

<u>*NOTE:*</u> The survival rates (of which there is one less than the actual number of survival rates since no animal survives beyond the 15 – 18 age group) lie along the **super diagonal** that is immediately above the main diagonal of the matrix.

Using this matrix, we can find other population distributions at the end of other cycles.

$P_1 = P_0 L$ $\qquad$ $P_2 = P_1 L = (P_0 L)L = P_0 L^2$ $\qquad$ In general, $P_k = P_0 L^k$

<u>Example 1:</u>

Find the population distribution of the above example after 24 months (8 cycles).

Suppose that the natural range for rats can sustain a pack that contains a maximum of 100 females. How long before this pack size is reached?